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# HYBRID EXPANSION METHOD FOR FREQUENCY RESPONSES AND THEIR SENSITIVITIES, PART I: UNDAMPED SYSTEMS

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Frequency responses and their sensitivities have been broadly applied to the areas of finite element model updating, structural damage detection, structural dynamic optimization and so on. A modal acceleration method for the frequency responses and a double-modal acceleration method for their sensitivities of undamped systems are derived in this paper. The two methods are based on the hybrid expansion, power series expansion and modal superposition, of the dynamic flexible matrix. Three steps are required to calculate the sensitivities using the proposed method. Firstly, frequency responses of a system excited by external forces are calculated by using modal acceleration. A pseudo-force vector is then computed from the product of the sensitivity matrix and the frequency response vector. Finally, a second-modal acceleration is applied to obtain the general frequency responses, that is, the sensitivities, under the pseudo-forces. Two modal truncation schemes, middle–high–modal and low–high–modal truncation schemes, are presented according to the values of the excited frequencies. The modal truncated errors of the frequency responses and their sensitivities will be reduced quickly when the two-modal acceleration methods are adopted. Although only the frequency responses and their sensitivities are discussed in this paper, the proposed methods are also valid for the frequency response functions, responses in time domain and their sensitivities. The results of a two-dimensional frame show that the proposed modal acceleration methods are efficient, especially for the sensitivities.



## 1. INTRODUCTION

Sensitivity analysis with respect to design parameters was first applied to optimal control and automated structural optimization in which gradient methods were used to find the search directions for optimum solution. More recently, the sensitivities of dynamic properties have been applied to finite element model updating, structural damage detection, dynamic optimization and so on. The sensitivities of eigenvalues and their corresponding eigenvectors of a structure have been discussed in detail during the past 30 years. The results of such work are fruitful and almost conclusive. However, there seems to be far less work

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being done directly on the frequency responses, which have even more practical applications.

Generally, there are two kinds of methods, i.e., direct method and modal superposition method, for calculating frequency responses. The direct approach is based on the direct frequency solution and results in an exact calculation of frequency responses. In this method, the decomposition of the system dynamic matrix, forward and back substitution processes are involved for every excited frequency. Hence, it is computationally very expensive when the number of the degrees of freedom and excited frequencies is large. Another disadvantage is its inability in handling modal damping, which is a vital concern in some applications.

The modal superposition method can be classified into several approaches: typical modal superposition [1], modified modal superposition [1] and modal acceleration [2, 3]. This kind of method can make up for the two drawbacks of the direct method. However, the eigenvalues and their corresponding eigenvectors of a system should be available. When modal truncation is adopted, the truncated errors of the frequency responses obtained from the modal acceleration approaches are very small compared with the typical and modified modal superposition approaches.

Three kinds of methods, direct approach, modal superposition approach and double-modal superposition approach, are usually used to calculate the sensitivities of the frequency responses. Similarly, the direct approach [4, 5] has two main shortcomings stated above. In the modal superposition method [6, 7], the sensitivities are obtained by taking the derivatives of frequency responses expressed in the modal superposition form. It is unnecessary to decompose the system dynamic matrix for every excited frequency in this method. However, a set of eigenpairs and their sensitivities are required. In addition, when there are repeated modes existing among the interested frequency range, the method may fail to obtain the correct results [7].

In 1993, Ting [8] proposed an improved method for calculating the sensitivities of frequency responses. It combines the modal acceleration method with the Ritz minimization technique to improve the modal approximation accuracy.

Based on the power-series expansion and modal superposition of the dynamic flexible matrix, a modal acceleration method and a double-modal acceleration method for frequency responses and their sensitivities of undamped system are derived respectively. The modal truncated errors of the frequency responses and their sensitivities will be reduced quickly when the two-modal acceleration methods are adopted. According to the values of the excited frequencies, two-modal truncation schemes, middle–high–modal and low–high–modal truncation schemes, are presented. When the frequencies of excited forces lie in the lower frequency range of a system, the middle–high–modal truncation scheme is applied. Similarly, the low–high–modal truncation scheme is used for the frequency responses and their sensitivities in the middle frequency range. The proposed methods are also valid for frequency response functions, responses in time domain and their sensitivities. A two-dimensional frame is applied to show the efficiency of the proposed methods.

## 2. THEORETICAL BACKGROUND

The dynamic equations of the $n$-degree-of-freedom undamped system can be written in a matrix form as

$$M(p)\ddot{X}(p,t) + K(p)X(p,t) = F(t), \tag{1}$$

where $M(p)$ and $K(p) \in R^{n\times n}$ are real symmetric mass and stiffness matrices respectively. They are the functions of design parameter vector $p$. For simplicity, this indication will be omitted in further discussion. $X(p,t)$ and $F(t) \in R^{n\times 1}$ are the displacement and excited force vectors respectively. A dot denotes one differentiation with respect to time $t$.

Suppose that all the components of vectors $X(t)$ and $F(t)$ are Fourier transformable and their transformations are $X(\omega)$ and $F(\omega)$ respectively. Assuming

$$\dot{X}(t) = X(t) = 0 \tag{2}$$

for $t = 0$, the Fourier transformation of equation (1) is

$$(K - \omega^2 M)X(\omega) = F(\omega), \tag{3}$$

where $\omega$ is circular frequency of excited forces. Hence, the frequency responses are

$$X(\omega) = (K - \omega^2 M)^{-1} F(\omega). \tag{4}$$

The sensitivities of the frequency responses can be obtained by taking the first partial derivative of equation (3) with respect to a selected design variable $p_j$ $(j = 1, 2, \ldots, m)$, that is

$$(K - \omega^2 M)\frac{\partial X(\omega)}{\partial p_j} = R(\omega), \tag{5}$$

where pseudo-force vector $R(\omega)$ is defined as

$$R(\omega) = S(\omega)X(\omega),\ S(\omega) = -\left(\frac{\partial K}{\partial p_j} - \omega^2\frac{\partial M}{\partial p_j}\right). \tag{6}$$

Hence, the sensitivities of the frequency responses are

$$\frac{\partial X(\omega)}{\partial p_j} = (K - \omega^2 M)^{-1}\ R(\omega). \tag{7}$$

Assuming that the eigenvalue and eigenvector matrices of the system are $\Lambda$ and $\Phi \in R^{n\times n}$, and

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \quad (\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n), \quad \Phi = [\phi_1\ \phi_2 \ldots \phi_n], \tag{8}$$

where $\lambda_i$ and $\phi_i$ are the $i$th eigenvalue and eigenvector. $\Lambda$ and $\Phi$ satisfy the following eigen-equation and orthogonalities:

$$K\Phi = M\Phi\Lambda, \tag{9}$$

$$\Phi^{\mathrm{T}} K\Phi = \Lambda, \tag{10}$$

$$\Phi^{\mathrm{T}} M\Phi = I, \tag{11}$$

where superscript T denotes matrix transpose. $I$ is an identity matrix of $n \times n$. From equations (9)–(11) one obtains

$$K^{-1} = \Phi\Lambda^{-1}\Phi^{\mathrm{T}}, \tag{12}$$

$$(K - \omega^2 M)^{-1} = \Phi(\Lambda - \omega^2 I)^{-1}\Phi^{\mathrm{T}}. \tag{13}$$

### 2.1. MODAL SUPERPOSITION METHOD FOR FREQUENCY RESPONSES

Substituting equation (13) into equation (4), one has

$$X(\omega) = \Phi(\Lambda - \omega^2 I)^{-1}\Phi^{\mathrm{T}} F(\omega). \tag{14}$$

The frequency responses, which are expressed in modal parameters in equation (14), can be expanded in modal space as

$$X(\omega) = \sum_{r=1}^{n} \frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2}\, \phi_r. \tag{15}$$

Equations (14) and (15) are the basic equations of frequency responses.

The excited frequencies can be classified into three categories compared with the natural frequencies of the system. (a) The excited frequencies are all very low and the largest one is still lower than the lowest natural frequency of the system. For this case, the calculation of the frequency responses and their sensitivities is very simple and will not be discussed in the following. (b) The excited frequencies are low and lie in the lower frequency range of the system. (c) The excited frequencies are a little high and lie in the middle frequency range. According to the division, the modal truncation can be divided into middle–high–modal truncation and low–high–modal truncation.

In the middle–high–modal truncation approach, both the middle and the higher modes of the system are truncated. Hence, only the modes in the lower frequency range are used to calculate the frequency responses and their sensitivities. Suppose that the lower $L$ modes are selected when modal truncation is applied; the frequency responses defined in equation (15) become

$$X_1^l(\omega) = \sum_{r=1}^{L} \frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2}\, \phi_r. \tag{16}$$

When the excited frequencies lie in the middle frequency range of the system, the number of the kept modes will be very large if equation (16) is still used to calculate the frequency responses. This makes it difficult to solve the eigenproblem (9). Hence, the low–high–modal truncation approach is applied. If the $L_1$th through $L_2$th modes are selected as the kept modes, the frequency responses can be expressed as

$$X_1^m(\omega) = \sum_{r=L_1}^{L_2} \frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2} \phi_r. \tag{17}$$

The modal truncated error of the frequency responses resulting from equations (16) and (17) are

$$E_1^l(\omega) = \sum_{r=L+1}^{n} \frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2} \phi_r, \tag{18}$$

$$E_1^m(\omega) = \sum_{r=1}^{L_1-1} \frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2} \phi_r + \sum_{r=L_2+1}^{n} \frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2} \phi_r. \tag{19}$$

The superscript $l$ and $m$ in equations (16–19) denote the frequency responses in the lower and higher frequency range respectively.

### 2.2. DOUBLE-MODAL SUPERPOSITION METHOD FOR THE SENSITIVITIES

Introducing equation (13) into equation (7) yields

$$\frac{\partial X(\omega)}{\partial p_j} = \Phi(\Lambda - \omega^2 I)^{-1} \Phi^{\mathrm{T}} R(\omega). \tag{20}$$

For the double-modal superposition method, three steps are required to calculate the sensitivities of frequency responses. (i) Compute the frequency responses of the system under excited forces. (ii) Calculate the pseudo-force vector $R(\omega)$ by using equation (6). (iii) After substituting $R(\omega)$ into equation (20), the sensitivities are obtained by a second-modal superposition, i.e.,

$$\frac{\partial X(\omega)}{\partial p_j} = \sum_{r=1}^{n} \frac{\phi_r^{\mathrm{T}} R(\omega)}{\lambda_r - \omega^2} \phi_r. \tag{21}$$

When the middle–high–modal truncation and low–high–modal truncation are applied, the sensitivities expressed in equation (21) can be rewritten as

$$\left(\frac{\partial X(\omega)}{\partial p_j}\right)_1^l = \sum_{r=1}^{L} \frac{\phi_r^{\mathrm{T}} R_1^l(\omega)}{\lambda_r - \omega^2} \phi_r,$$

$$R_1^l(\omega) = S(\omega) X_1^l(\omega), \tag{22}$$

$$\left(\frac{\partial X(\omega)}{\partial p_j}\right)_1^m = \sum_{r=L_1}^{L_2} \frac{\phi_r^{\mathrm{T}} R_1^m(\omega)}{\lambda_r - \omega^2} \phi_r,$$

$$R_1^m(\omega) = S(\omega) X_1^m(\omega), \tag{23}$$

where $X_1^l(\omega)$ and $X_1^m(\omega)$ are defined by equations (16) and (17). Equations (22) and (23) are the governing equations of the sensitivities of the frequency responses in the lower and higher frequency range respectively.

The modal truncated errors of the sensitivities resulting from equations (22) and (23) are

$$\bar{E}_1^l(\omega) = \sum_{r=1}^{n} \frac{\phi_r^{\mathrm{T}} S(\omega) E_1^l(\omega)}{\lambda_r - \omega^2}\, \phi_r + \sum_{r=L+1}^{n} \frac{\phi_r^{\mathrm{T}} R_1^l(\omega)}{\lambda_r - \omega^2}\, \phi_r, \tag{24}$$

$$\bar{E}_1^m(\omega) = \sum_{r=1}^{n} \frac{\phi_r^{\mathrm{T}} S(\omega) E_1^m(\omega)}{\lambda_r - \omega^2}\, \phi_r + \sum_{r=1}^{L_1-1} \frac{\phi_r^{\mathrm{T}} R_1^m(\omega)}{\lambda_r - \omega^2}\, \phi_r + \sum_{r=L_2+1}^{n} \frac{\phi_r^{\mathrm{T}} R_1^m(\omega)}{\lambda_r - \omega^2}\, \phi_r \tag{25}$$

respectively. Obviously, the errors of the sensitivities of frequency responses are composed of two parts. One results from the truncated errors of frequency responses, which is expressed by the first part in equations (24) and (25). The other results from the modal truncation when the modal superposition method is used to calculate the sensitivities, which is denoted by the residual parts in the two equations.

## 3. FREQUENCY RESPONSES AND THEIR SENSITIVITIES IN LOWER FREQUENCY RANGE

### 3.1. MODAL ACCELERATION METHOD FOR FREQUENCY RESPONSES

It can be proven that the inverse of matrix $(\Lambda - \omega^2 I)$ in equation (13) can be expanded in a power series as

$$(\Lambda - \omega^2 I)^{-1} = \Lambda^{-1} \sum_{h=0}^{H} (\omega^2 \Lambda^{-1})^h + (\omega^2 \Lambda^{-1})^{H+1} (\Lambda - \omega^2 I)^{-1}, \tag{26}$$

where $H$ is any integer that is larger than $-1$. $H = -1$ indicates that no power series is adopted. Substituting equation (26) into equation (14), the frequency responses can be expressed as

$$X(\omega) = X_A(\omega) + X_S(\omega), \tag{27}$$

where $X_A(\omega)$ and $X_S(\omega)$ denote the frequency responses defined by the summation of the former $H + 1$ items and by the residue of the power series respectively. They are

$$X_A(\omega) = \Phi \Lambda^{-1} \sum_{h=0}^{H} (\omega^2 \Lambda^{-1})^h \Phi^{\mathrm{T}} F(\omega), \tag{28}$$

$$X_S(\omega) = \Phi (\omega^2 \Lambda^{-1})^{H+1} (\Lambda - \omega^2 I)^{-1} \Phi^{\mathrm{T}} F(\omega). \tag{29}$$

The frequency responses $X(\omega)$ are divided into two parts, $X_A(\omega)$ and $X_S(\omega)$, only because the two parts are associated with the modal acceleration and modal superposition respectively.

Firstly, the frequency responses described in equation (28) are discussed. It can be seen from equation (28) that the frequency responses $X_A(\omega)$ are expressed with modal parameters of the system. However, almost all the higher eigenvalues and eigenvectors are usually not available for a large and/or complex system. Hence, it is necessary to rewrite them with physical parameters of the system. Using equation (12), the matrix $K^{-1}(\omega^2 MK^{-1})^h$ can be expanded as

$$K^{-1}(\omega^2 MK^{-1})^h = \Phi\Lambda^{-1}\Phi^{\mathrm{T}} \underbrace{(M\Phi\omega^2\Lambda^{-1}\Phi^{\mathrm{T}})\cdots(M\Phi\omega^2\Lambda^{-1}\Phi^{\mathrm{T}})}_{h}. \tag{30}$$

Considering the mass matrix normalization of the eigenvector matrix $\Phi$, equation (30) is simplified as

$$K^{-1}(\omega^2 MK^{-1})^h = \Phi\Lambda^{-1}(\omega^2\Lambda^{-1})^h\Phi^{\mathrm{T}}. \tag{31}$$

Introducing equation (31) into equation (28) yields

$$X_A(\omega) = K^{-1}\sum_{h=0}^{H}(\omega^2 MK^{-1})^h F(\omega). \tag{32}$$

Obviously, the parameters on the right-hand of equation (32) are all known in advance.

The frequency responses $X_S(\omega)$ defined by the residue of the power series can be rewritten as

$$X_S(\omega) = \Phi G(\omega)\Phi^{\mathrm{T}} F(\omega), \tag{33}$$

where $G(\omega)$ is a diagonal matrix and the $r$th diagonal element is

$$g_r = \left(\frac{\omega^2}{\lambda_r}\right)^{H+1}\frac{1}{\lambda_r - \omega^2} \quad (r = 1, 2, \ldots, n). \tag{34}$$

Equation (33) can be expanded in modal space as

$$X_S(\omega) = \sum_{r=1}^{n}\left(\frac{\omega^2}{\lambda_r}\right)^{H+1}\frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2}\,\phi_r. \tag{35}$$

Substituting equations (32) and (35) into the right-hand side of equation (27), the frequency responses are obtained as

$$X(\omega) = K^{-1}\sum_{h=0}^{H}(\omega^2 MK^{-1})^h F(\omega) + \sum_{r=1}^{n}\left(\frac{\omega^2}{\lambda_r}\right)^{H+1}\frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2}\,\phi_r. \tag{36}$$

Assuming the lower $L$ modes are selected as the kept modes when modal truncation is adopted, the frequency responses can be expressed as

$$X_2^l(\omega) = K^{-1} \sum_{h=0}^{H} (\omega^2 M K^{-1})^h F(\omega) + \sum_{r=1}^{L} \left(\frac{\omega^2}{\lambda_r}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2} \phi_r. \tag{37}$$

The modal truncated errors resulting from equation (37) are

$$E_2^l(\omega) = \sum_{r=L+1}^{n} \left(\frac{\omega^2}{\lambda_r}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2} \phi_r. \tag{38}$$

### 3.2. DOUBLE-MODAL ACCELERATION METHOD FOR THE SENSITIVITIES

After the frequency responses are obtained, they can be used to calculate their sensitivities. Substituting equation (26) into equation (20), one has

$$\frac{\partial X(\omega)}{\partial p_j} = \left(\frac{\partial X(\omega)}{\partial p_j}\right)_A + \left(\frac{\partial X(\omega)}{\partial p_j}\right)_S, \tag{39}$$

$$\left(\frac{\partial X(\omega)}{\partial p_j}\right)_A = \Phi \Lambda^{-1} \sum_{h=0}^{H} (\omega^2 \Lambda^{-1})^h \Phi^{\mathrm{T}} R(\omega), \tag{40}$$

$$\left(\frac{\partial X(\omega)}{\partial p_j}\right)_S = \Phi (\omega^2 \Lambda^{-1})^{H+1} (\Lambda - \omega^2 I)^{-1} \Phi^{\mathrm{T}} R(\omega). \tag{41}$$

Based on the same derivative procedure of equation (36), we have

$$\frac{\partial X(\omega)}{\partial p_j} = K^{-1} \sum_{h=0}^{H} (\omega^2 M K^{-1})^h R(\omega) + \sum_{r=1}^{n} \left(\frac{\omega^2}{\lambda_r}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} R(\omega)}{\lambda_r - \omega^2} \phi_r. \tag{42}$$

When the lower $L$ modes are selected as the kept modes, the sensitivities of frequency responses are obtained by the double-modal acceleration as

$$\begin{aligned} \left(\frac{\partial X(\omega)}{\partial p_j}\right)_2^l &= K^{-1} \sum_{h=0}^{H} (\omega^2 M K^{-1})^h R_2^l(\omega) + \sum_{r=1}^{L} \left(\frac{\omega^2}{\lambda_r}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} R_2^l(\omega)}{\lambda_r - \omega^2} \phi_r, \\ R_2^l(\omega) &= S(\omega) X_2^l(\omega), \end{aligned} \tag{43}$$

where $S(\omega)$ is defined in equation (6).

Similarly, three steps are required to calculate the sensitivities of frequency responses by using the double-modal acceleration method. (i) The frequency responses of the system under excited forces are computed by using the modal acceleration method in equation (37). (ii) Pseudo-force vector $R_2^l(\omega)$ is calculated by using the second equation of equation (43). (iii) $R_2^l(\omega)$ is substituted into the first equation of equation (43), and then the sensitivities are obtained by a second-modal acceleration.

The truncated errors resulting from equation (43) are

$$\bar{E}_2^l(\omega) = K^{-1} \sum_{h=0}^{H} (\omega^2 M K^{-1})^h S(\omega) E_2^l(\omega)$$

$$+ \sum_{r=1}^{n} \left(\frac{\omega^2}{\lambda_r}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} S(\omega) E_2^l(\omega)}{\lambda_r - \omega^2} \phi_r + \sum_{r=L+1}^{n} \left(\frac{\omega^2}{\lambda_r}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} R_2^l(\omega)}{\lambda_r - \omega^2} \phi_r. \quad (44)$$

The errors of the sensitivities are composed of two parts. The structure is similar to that of equation (24). One results from the truncated errors of the frequency responses calculated with equation (37) and is expressed by the former two parts in equation (44). The other results from the modal truncation when calculating the sensitivities themselves and is denoted by the third part in equation (44).

### 3.3 CONVERGENT CONDITION

Equation (38) has a coefficient $(\omega^2/\lambda_r)^{H+1}$ compared with equation (18). In order that the truncated errors of the frequency responses obtained from equation (37) are reduced with the increase of the items $H$ of the power series, the equation

$$\left|\frac{\omega^2}{\lambda_r}\right| < 1 \quad (r > L) \quad (45)$$

should be satisfied for the truncated modes. Considering equation (8), one has

$$\omega_{max}^2 < \lambda_{L+1}, \quad (46)$$

where $\omega_{max}$ is the highest frequency among the excited frequencies. Usually, one or two more modes are selected to make the covergence faster. Obviously, when conditions (45) or (46) is satisfied, the modal acceleration method can make the approximate frequency responses $X_2^l(\omega)$ very close to the real $X(\omega)$, which leads to $|E_2^l(\omega)| \ll |X_2^l(\omega)|$. Similarly, when condition (46) is satisfied, the errors of the sensitivities of frequency responses are convergent too. Hence, equation (46) is the convergent condition of the frequency responses and their sensitivities.

## 4. FREQUENCY RESPONSES AND THEIR SENSITIVITIES IN MIDDLE FREQUENCY RANGE

### 4.1. MODAL ACCELERATION FOR FREQUENCY RESPONSES

Considering the eigenvalue shifting technique, we have

$$(K - \omega^2 M) = (\bar{K} - \bar{\omega}^2 M), \quad (47)$$

where

$$\bar{K} = K - qM, \quad \bar{\omega}^2 = \omega^2 - q. \tag{48}$$

Usually, the eigenvalue shifting $q$ is

$$q \approx \frac{\omega_{min}^2 + \omega_{max}^2}{2} \tag{49}$$

and should satisfy $q \neq \lambda_r$ $(r = 1, 2, \ldots, n)$. $\omega_{min}$ and $\omega_{max}$ are the under and upper boundary values of the excited frequencies. Substituting equation (47) into equation (4), the frequency responses are obtained as

$$X(\omega) = (\bar{K} - \bar{\omega}^2 M)^{-1} F(\omega). \tag{50}$$

Equation (50) can be rewritten as

$$X(\omega) = \Phi(\bar{\Lambda} - \bar{\omega}^2 I)^{-1} \Phi^{\mathrm{T}} F(\omega) \tag{51}$$

by using the modal parameters of the system, where

$$\bar{\Lambda} = \Lambda - qI. \tag{52}$$

When modal acceleration is applied, the frequency responses can be expressed as

$$X(\omega) = \bar{K}^{-1} \sum_{h=0}^{H} (\bar{\omega}^2 M \bar{K}^{-1})^h F(\omega) + \sum_{r=1}^{n} \left(\frac{\omega^2 - q}{\lambda_r - q}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2} \phi_r. \tag{53}$$

Assume that the $L_1$th through the $L_2$th modes are selected as the kept modes when modal truncation is applied. The frequency responses in the middle frequency range of the system are

$$X_2^m(\omega) = \bar{K}^{-1} \sum_{h=0}^{H} (\bar{\omega}^2 M \bar{K}^{-1})^h F(\omega) + \sum_{r=L_1}^{L_2} \left(\frac{\omega^2 - q}{\lambda_r - q}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2} \phi_r. \tag{54}$$

The errors resulting from equation (54) are

$$E_2^m(\omega) = \sum_{r=1}^{L_1-1} \left(\frac{\omega^2 - q}{\lambda_r - q}\right) \frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2} \phi_r + \sum_{r=L_2+1}^{n} \left(\frac{\omega^2 - q}{\lambda_r - q}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} F(\omega)}{\lambda_r - \omega^2} \phi_r. \tag{55}$$

### 4.2. DOUBLE-MODAL ACCELERATION METHOD FOR THE SENSITIVITIES

Substituting equation (47) into equation (7) yields

$$\frac{\partial X(\omega)}{\partial p_j} = (\bar{K} - \bar{\omega}^2 M)^{-1} R(\omega). \tag{56}$$

Based on the similar derivation above, one has

$$\frac{\partial X(\omega)}{\partial p_j} = \bar{K}^{-1} \sum_{h=0}^{H} (\bar{\omega}^2 M \bar{K}^{-1})^h R(\omega) + \sum_{r=1}^{n} \left(\frac{\omega^2 - q}{\lambda_r - q}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} R(\omega)}{\lambda_r - \omega^2} \phi_r. \tag{57}$$

When modal truncation is applied, the sensitivities are

$$\left(\frac{\partial X(\omega)}{\partial p_j}\right)_2^m = \bar{K}^{-1} \sum_{h=0}^{H} (\bar{\omega}^2 M \bar{K}^{-1})^h R_2^m(\omega) + \sum_{r=L_1}^{L_2} \left(\frac{\omega^2 - q}{\lambda_r - q}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} R_2^m(\omega)}{\lambda_r - \omega^2} \phi_r,$$

$$R_2^m(\omega) = S(\omega) X_2^m(\omega). \tag{58}$$

The errors resulting from equation (58) are

$$\begin{aligned}\bar{E}_2^m(\omega) = {} & \bar{K}^{-1} \sum_{h=0}^{H} (\omega^2 M \bar{K}^{-1})^h S(\omega) E_2^m(\omega) + \sum_{r=1}^{n} \left(\frac{\omega^2 - q}{\lambda_r - q}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} S(\omega) E_2^m(\omega)}{\lambda_r - \omega^2} \phi_r \\ & + \sum_{r=1}^{L_1-1} \left(\frac{\omega^2 - q}{\lambda_r - q}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} R_2^m(\omega)}{\lambda_r - \omega^2} \phi_r + \sum_{r=L_2+1}^{n} \left(\frac{\omega^2 - q}{\lambda_r - q}\right)^{H+1} \frac{\phi_r^{\mathrm{T}} R_2^m(\omega)}{\lambda_r - \omega^2}.\end{aligned} \tag{59}$$

### 4.3. CONVERGENT CONDITION

In order that the frequency responses obtained from the modal acceleration method are more accurate than those from the typical modal superposition method and that the truncated errors will decrease with the increase of the items $H$ of the power series, the selected $L_1$ and $L_2$ should satisfy

$$\left|\frac{\omega^2 - q}{\lambda_r - q}\right| < 1 \quad (r \leqslant L_1 - 1, r \geqslant L_2 + 1). \tag{60}$$

By considering equation (49), one obtains

$$\lambda_{L_1-1} < \omega_{min}^2, \quad \lambda_{L_2+1} > \omega_{max}^2. \tag{61}$$

Equation (61) is the governing equation of $L_1$ and $L_2$. It means that the frequencies corresponding to the truncated modes should lie outside of the excited frequency range. Usually, one or two more modes are selected to make the covergence faster. When equation (61) is satisfied, the truncated errors of the sensitivities will decrease with the increase of $H$.

## 5. NUMERICAL EXAMPLE

A two-dimensional frame shown in Figure 1 is considered here. The height and width of each story are 2·0 and 4·0 m respectively. It has a total of 32 nodes and 96 degrees of freedom. For all the beams, modulus of elasticity = 2·1E11 N/m$^2$, mass density = 7830 kg/m$^3$, area moment of inertia = 8·0E-9 m$^4$, cross-sectional area = 2·4E-4 m$^2$. The former 20 natural frequencies are listed in Table 1.

### 5.1. FREQUENCY RESPONSES

Assume that an identity force is located at node 28 in the $y$ direction. The frequency range of the force is 0–60 rad/s. It lies in the lower frequency range of

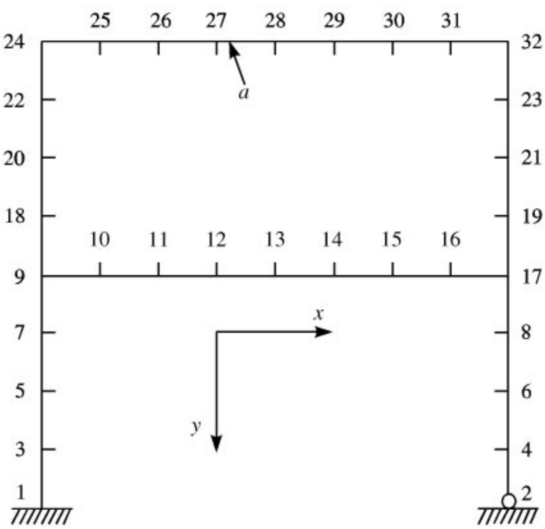


Figure 1. The schematic of a two-dimensional frame.

TABLE 1

*The former 20 natural frequencies of the frame* (rad/s)

| Order | Frequency | Order | Frequency | Order | Frequency | Order | Frequency |
|---|---|---|---|---|---|---|---|
| 1 | 6·37976 | 6 | 81·1249 | 11 | 180·054 | 16 | 366·699 |
| 2 | 14·7552 | 7 | 97·1531 | 12 | 199·266 | 17 | 417·440 |
| 3 | 28·4262 | 8 | 110·864 | 13 | 214·255 | 18 | 439·016 |
| 4 | 30·6140 | 9 | 140·522 | 14 | 309·187 | 19 | 476·141 |
| 5 | 35·9307 | 10 | 151·388 | 15 | 341·922 | 20 | 520·263 |

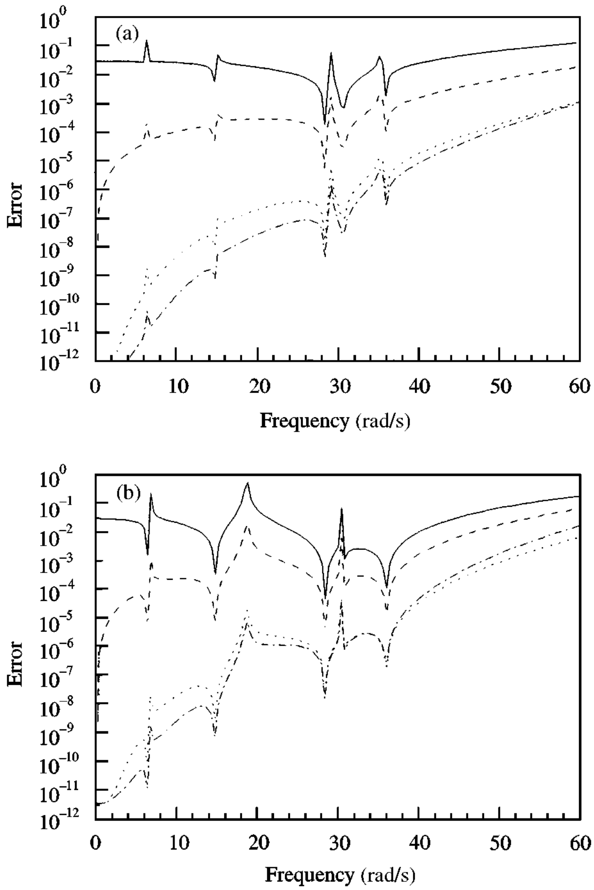


Figure 2. Errors of frequency responses in lower frequency range at (a) node (28); (b) node (13). ——, $H = -1$; – – –, $H = 0$; . . . ., $H = 2$; –·–·–, $H = 4$.

the structure. According to the frequency characteristics of the excited force and equation (46), $q = 0$ and $L = 5$ are selected. This means that all the modes which are higher than the 5th are truncated when modal truncation is adopted. The frequency responses at node 28 and 13 in the $y$ direction for various $H$ are calculated. The Errors of the approximate frequency responses are shown in Figures 2(a) and 2(b). The Error is defined as

$$\text{Error} = |(x_{appro} - x_{exact})/x_{exact}|, \tag{62}$$

where $x_{appro}$ and $x_{exact}$ denote the approximate and exact frequency response respectively. In the two figures, $H = -1$ denotes the Errors of the frequency responses obtained from equation (16).

The accuracy of frequency responses obtained from equation (16) is very low. The largest percent errors (= 100Errors%) in Figures 2(a) and 2(b), for example, are 16·02% and 53·22% respectively. After the modal acceleration method is applied, the accuracy increases quickly. For the cases of $H = 0$, 2 and 4 in

Figures 2(a) and 2(b), the percent errors, which correspond to the largest errors for $H = -1$, are 0·0748, 0·0000, 0·0000% and 1·976, 0·0020, 0·0007% respectively. The accuracy of the frequency responses for lower frequencies increases much more quickly than that for higher frequencies. The Error at 10·0 rad/s for $H = -1$ is about $10^7$ times as large as that for $H = 4$. However, it is about 100 times at 60·0 rad/s. The reason can be explained from equation (38).

Assume the frequency range of the excited force is 120–190 rad/s. It lies in the middle frequency range of the structure. According to the frequency characteristics of the excited force and equation (61), $q = 27\,200$, $L_1 = 8$ and $L_2 = 13$ are selected. This means that the former seven modes and all the modes which are higher than the 13th are truncated when modal truncation is adopted. The frequency responses at nodes 28 and 13 in the $y$ direction for various $H$ are calculated. The Errors of these approximate frequency responses are shown in Figures 3(a) and 3(b). In the two figures, $H = -1$ denotes the Errors of the approximate frequency responses obtained from equation (17).

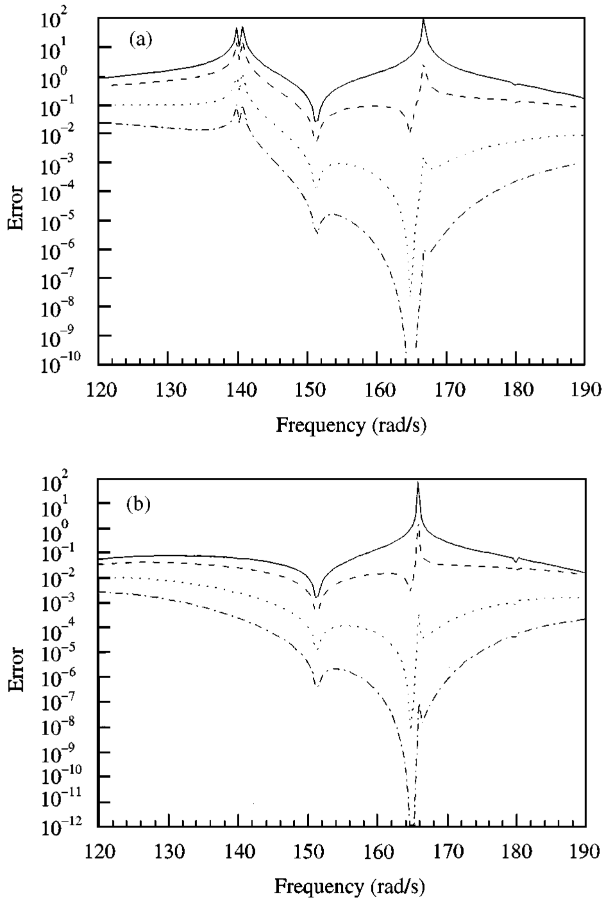


Figure 3. Errors of frequency responses in middle frequency range at (a) node 28; (b) node 13. ——, $H = -1$; – – –, $H = 0$; . . . . ., $H = 2$; – · – · –, $H = 4$.

The accuracy of frequency responses obtained from equation (17) is very low. The largest percent errors in Figures 3(a) and 3(b), for example, are 9880 and 7463% respectively. Obviously, these frequency responses are useless. After the modal acceleration method is applied, the accuracy increases very quickly. The percent errors, which correspond to the largest errors for $H = -1$, for $H = 0$, 2 and 4 in Figures 3(a) and 3(b), are 286·6, 0·1579, 0·0000% and 174·4, 0·0362, 0·0000% respectively. The accuracy of the frequency responses for middle frequencies increases much more quickly than that for lower and higher frequencies.

## 5.2. SENSITIVITIES OF FREQUENCY RESPONSES

Assume that the modulus of elasticity of element "a" in Figure 1 is selected as the design parameter. The sensitivities of the frequency responses discussed above are calculated. Their Errors are shown in Figures 4 and 5 respectively. The accuracy of the sensitivities obtained from the double-modal superposition method,

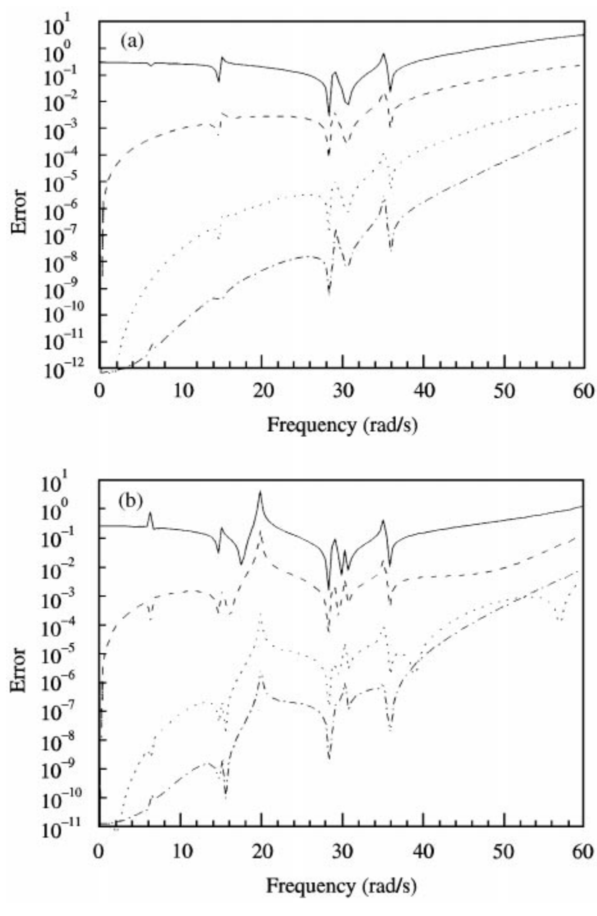


Figure 4. Errors of sensitivities in lower frequency range at (a) node 28; (b) node 13. ——, $H = -1$; – – –, $H = 0$; . . . . ., $H = 2$; – · – · – ·, $H = 4$.

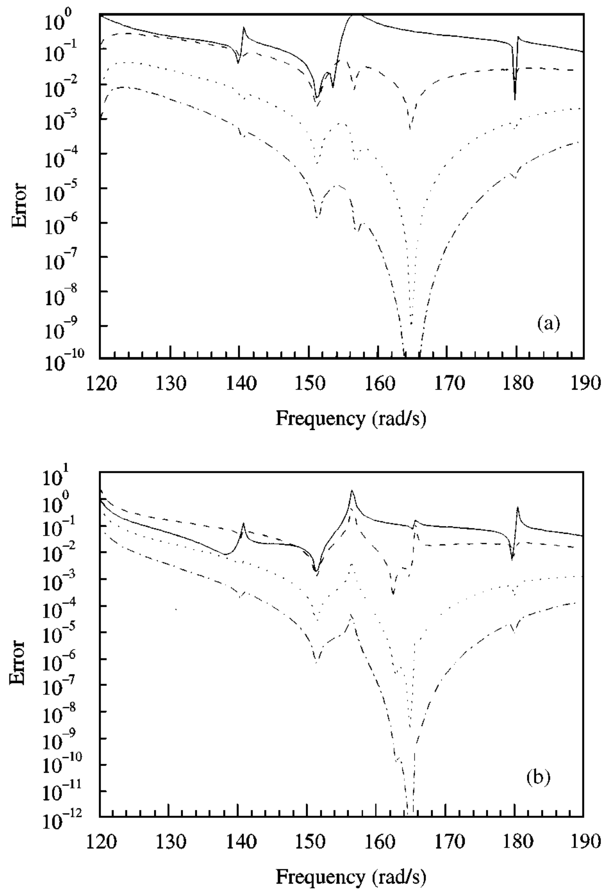


Figure 5. Errors of sensitivities in middle frequency range at (a) node 28; (b) node 13. ——, $H = -1$; – – –, $H = 0$; . . . . . , $H = 2$; – – – , $H = 4$.

i.e., equations (22) and (23), is very low. It increases quickly if the double-modal acceleration method is applied. When the former five items of the power series, that is $H = 4$, are adopted, the Errors are reduced by at least 1000 times.

For the proposed modal acceleration methods, 701 times steps, i.e., $\Delta t = 0{\cdot}1$ rad/s, are applied to calculate the frequency responses and their sensitivities in the middle frequency range. The computed time is 4·2 s in a PC-100. If the direct method is used for this problem, 701 times of decomposition, forward and back-substitutions of dynamic stiffness matrix $K - \omega^2 M$ are required. The corresponding computed time is 46·3 s. Obviously, the latter is much more computationally expensive than the former. If the number of time steps and degrees of freedom becomes larger, the computational time of the latter will increase rapidly. If the accuracy, which is equivalent to the present method for $H = 4$, is required for the modal superposition method, the former 25 frequencies and their corresponding mode shapes should be selected as the kept modes. This makes the computation of the eigen-problem much more expensive because the solution time rises drastically as the number of eigenpairs increases [9].

The Errors of frequency responses and their sensitivities for $H = -1$ in Figures 2–5 are redrawn in Figure 6. In these figures, FR and SFR denote frequency responses and their sensitivities respectively. If the same modes are adopted for calculating frequency responses and their sensitivities in the lower frequency range, the Errors of the former are generally much smaller than the latter. However, it is not true for them in the middle frequency range. This has mainly resulted from partial offset of the truncated errors of the sensitivities in two modal truncations.

## 6. CONCLUSIONS

Based on the hybrid expansion of the dynamic flexible matrix, a modal acceleration method and a double-modal acceleration method for frequency responses and their sensitivities of undamped systems are derived respectively. When the frequencies of excited forces lie in the lower frequency range of the system, both the middle and the higher modes can be truncated by using the

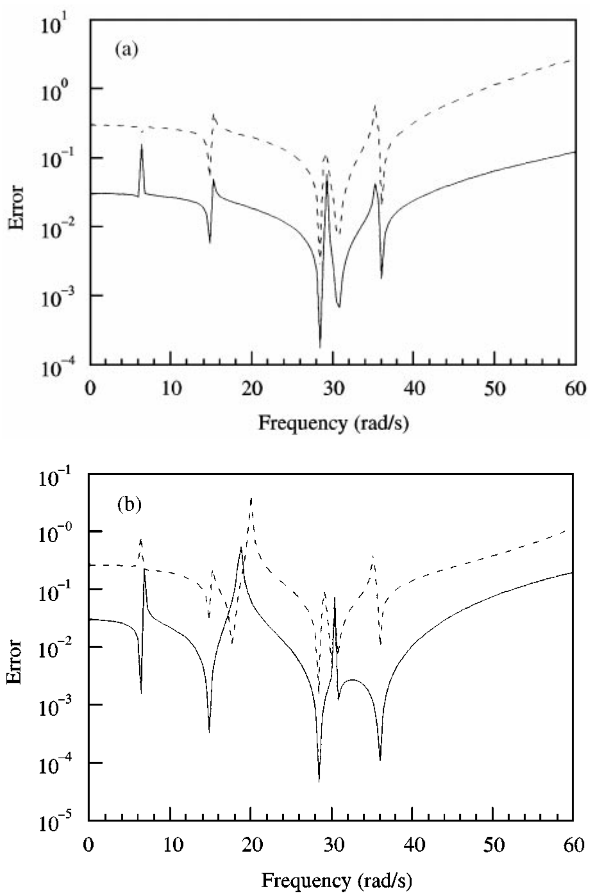


Figure 6. Comparison of the errors of frequency responses and their sensitivities: (a) errors in lower frequency range at node 28; (b) errors in lower frequency range at node 13; (c) errors in middle frequency range at node 28; (d) errors in middle frequency range at node 13. ——, FR; - - - - - -, SFR.

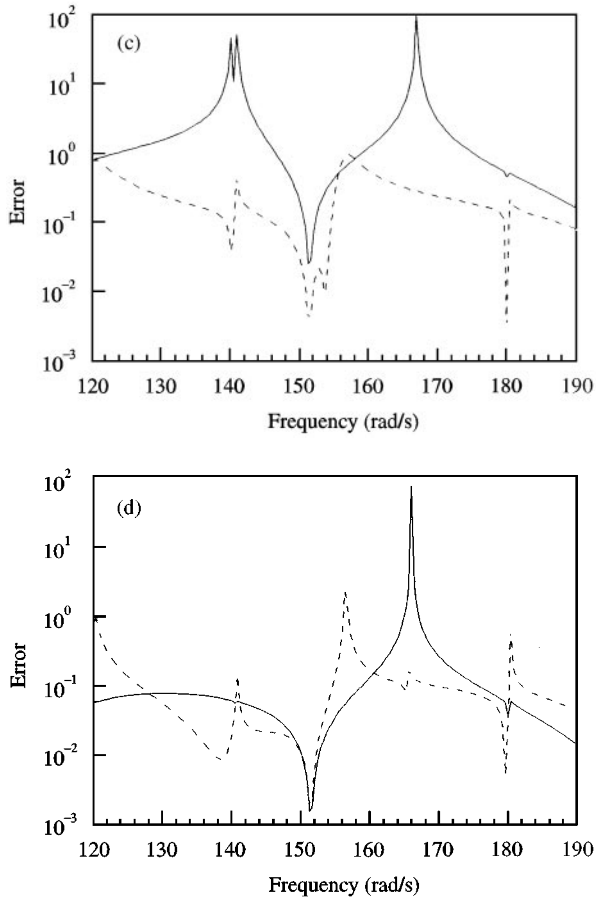


Figure 6. (*Continued*)

methods. When the excited frequencies lie in the middle frequency range, the higher as well as the lower modes can be truncated. The following five conclusions can be drawn from these methods.

(1) Theoretically, the natural frequencies corresponding to the truncated modes should lie outside of the frequency range of the excited forces when using the proposed methods. However, one or two more modes are selected as the kept modes to improve the convergent rate of the acceleration methods.
(2) The accuracy of frequency responses and their sensitivities obtained from modal superposition and double-modal superposition methods are very low. When the two-modal acceleration methods are adopted, the modal truncated errors are reduced very quickly, especially for the sensitivities. Generally, highly accurate results can be obtained when several items of the power series are adopted.
(3) When a system has rigid modes, $q \neq 0$ in equation (48) can make the stiffness matrix non-singular.

(4) The proposed methods are also valid for frequency response functions and their sensitivities, responses in time domain and their sensitivities.

(5) If the same modes are adopted for calculating frequency responses and their sensitivities in the lower frequency range, the Errors of the former are generally much smaller than the latter. However, it is not true for them in the middle frequency range. This has mainly resulted from partial offset of the truncated errors in two-modal truncations.

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## REFERENCES

1. R. R. Craig 1981 *Structural Dynamics*. New York: John Wiley.
2. M. A. Akgun 1993 *Journal of Sound and Vibration* **167**, 289–302. A new family of mode-superposition methods for response calculations.
3. Z.-Q. Qu and Z.-F. Fu 1998 *Chinese Journal of Computational Mechanics* **17**, 91–95. A highly power series expansion method for calculating frequency response function.
4. A. G. Nalecz and J. Wicker 1998 *Journal of Sound and Vibration* **120**, 517–526. Design sensitivity analysis of mechanical system in frequency domain.
5. T. Ting, T. L. C. Chen and W. J. Twomey 1992 *Finite Elements in Analysis and Design* **11**, 213–220. Finite element model refinement with actual forced response of structures.
6. R. S. Aharp and P. C. Brooks 1998 *Journal of Sound and Vibration* **126**, 167–172. Sensitivities of frequency response functions of linear dynamic systems to variations in design parameter values.
7. T. Y. Chen 1992 *International Journal for Numerical Methods in Engineering* **33**, 1927–1940. Optimum design of structures with both natural frequency and frequency response constraints.
8. T. Ting 1993 *AIAA Journal* **31**, 1965–1967. Design of sensitivity analysis of structural frequency response.
9. K. J. Bathe 1982 *Finite Element Procedures in Engineering Analysis*. Englewood Cliffs, NJ: Prentice-Hall.